

# Picone identities for half-linear elliptic operators with $p(x)$ -Laplacians and applications to Sturmian comparison theory <sup>☆</sup>

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## Abstract

Picone identities are established for a class of half-linear elliptic operators with  $p(x)$ -Laplacians, and Sturmian comparison theorems are obtained on the basis of the Picone identities. Generalizations to half-linear elliptic inequalities with mixed nonlinearities are discussed, and specializations to half-linear partial or ordinary differential inequalities with  $p(x)$ -Laplacians are shown.

*Key words:*  $p(x)$ -Laplacian, Picone identity, Picone-type inequality, half-linear, elliptic, Sturmian comparison theory

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## 1. Introduction

The operator  $-\nabla \cdot (|\nabla u|^{p(x)-2} \nabla u)$  is said to be  $p(x)$ -Laplacian, and becomes  $p$ -Laplacian  $-\nabla \cdot (|\nabla u|^{p-2} \nabla u)$  if  $p(x) = p$  (constant), where the dot  $\cdot$  denotes the scalar product,  $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$  and  $|x|$  denotes the Euclidean length of  $x \in \mathbb{R}^n$ . There has been much current interest in studying various mathematical problems with variable exponent growth condition. The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (cf. [20, 27]).

Existence of weak solutions of the elliptic equation with  $p(x)$ -Laplacian

$$-\nabla \cdot (a(x)|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = f(x, u) \quad \text{in } \mathbb{R}^n$$

were investigated by several authors, see, for example, [5, 7, 14, 25]. For the existence of weak solutions for  $p(x)$ -Laplacian Dirichlet problem, we refer to [8, 13, 15, 16].

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The paper [26] by Zhang seems to be the first paper dealing with oscillations of solutions of  $p(x)$ -Laplacian equations. In [26] oscillation problem for the  $p(t)$ -Laplacian equation

$$(|u'|^{p(t)-2}u')' + t^{-\theta(t)}g(t, u) = 0, \quad t > 0$$

was treated. Motivated by Zhang [26], we establish Picone identities and Sturmian comparison theorems for half-linear elliptic inequalities.

Sturmian comparison theorems for half-linear elliptic equations

$$\begin{aligned} \nabla \cdot (a(x)|\nabla u|^{\alpha-1}\nabla u) + c(x)|u|^{\alpha-1}u &= 0, \\ \nabla \cdot (A(x)|\nabla v|^{\alpha-1}\nabla v) + C(x)|v|^{\alpha-1}v &= 0, \end{aligned}$$

where  $\alpha > 0$ , were derived by utilizing a Picone identity, where we mean by *half-linear* that a solution multiplied by any constant is also a solution. We refer the reader to Allegretto [1], Allegretto and Huang [3, 4], Bognár and Došlý [6], Došlý [9], Dunninger [12], Kusano, Jaroš and Yoshida [19], Yoshida [21, 22, 23, 24] for Picone identities and Sturmian comparison theorems, and to Došlý [10], Došlý and Řehák [11] for half-linear ordinary differential equations.

It might be natural to consider more general elliptic equations

$$\begin{aligned} \nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u) + c(x)|u|^{\alpha(x)-1}u &= 0, \\ \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v) + C(x)|v|^{\alpha(x)-1}v &= 0, \end{aligned}$$

where  $\alpha(x) > 0$ , but the above equations are not half-linear if  $\alpha(x)$  is not a constant. In order to obtain some oscillation results such as Sturmian comparison theorems, etc., which are generalizations of those of linear differential equations, we first determine a class of half-linear elliptic equations with  $p(x)$ -Laplacians.

The objective of this paper is to establish Picone identities for half-linear elliptic inequalities

$$uq[u] \geq 0, \tag{1.1}$$

$$vQ[v] \leq 0, \tag{1.2}$$

where  $q$  and  $Q$  are defined by

$$\begin{aligned} q[u] &:= \nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u) - a(x)(\log |u|)|\nabla u|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla u \\ &\quad + |\nabla u|^{\alpha(x)-1}b(x) \cdot \nabla u + c(x)|u|^{\alpha(x)-1}u, \end{aligned} \tag{1.3}$$

$$\begin{aligned} Q[v] &:= \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v) - A(x)(\log |v|)|\nabla v|^{\alpha(x)-1}\nabla \alpha(x) \cdot \nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x) \cdot \nabla v + C(x)|v|^{\alpha(x)-1}v, \end{aligned} \tag{1.4}$$

and derive Sturmian comparison theorems for  $q$  and  $Q$  by using the Picone identities. In Section 2 we first show that (1.1) and (1.2) are *half-linear* in the sense that a constant multiple of a solution  $u$  [resp.  $v$ ] is also a solution of (1.1) [resp. (1.2)] (see Proposition 2.1), and then establish Picone identities for  $q$  and  $Q$ . We mention, in particular, the paper [2] by Allegretto in which Picone

Identity arguments are used, and the formulae that are closely related to Picone identities in Section 2 are established.

In Section 3 we derive Sturmian comparison theorems for  $q$  and  $Q$ , and Section 4 is devoted to specializations to the case  $\alpha(x) = \alpha > 0$ , and to half-linear ordinary differential equations with  $p(t)$ -Laplacians which seems to be unknown.

## 2. Picone identities

Let  $G$  be a bounded domain in  $\mathbb{R}^n$  with piecewise smooth boundary  $\partial G$ . It is assumed that  $a(x), A(x) \in C(\overline{G}; (0, \infty))$ ,  $b(x), B(x) \in C(\overline{G}; \mathbb{R}^n)$ ,  $c(x), C(x) \in C(\overline{G}; \mathbb{R})$ , and that  $\alpha(x) \in C^1(\overline{G}; (0, \infty))$ .

The domain  $\mathcal{D}_q(G)$  of  $q$  is defined to be the set of all functions  $u$  of class  $C^1(\overline{G}; \mathbb{R})$  such that  $a(x)|\nabla u|^{\alpha(x)-1}\nabla u \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$ . The domain  $\mathcal{D}_Q(G)$  of  $Q$  is defined similarly.

We note in (1.1) that  $\log |u|$  has singularities at zeros of  $u(x)$ , but  $u \log |u|$  is continuous at every zero  $x_0$  if we define  $u \log |u| = 0$  at  $x = x_0$ , in view of the fact that  $\lim_{\varepsilon \rightarrow +0} \varepsilon \log \varepsilon = 0$ . We make the similar remarks in (1.2).

We consider the elliptic inequalities

$$uq[u] \geq 0 \quad \text{in } G, \quad (2.1)$$

$$vQ[v] \geq 0 \quad \text{in } G, \quad (2.2)$$

where  $q$  and  $Q$  are defined by (1.3) and (1.4).

By a *solution*  $u$  [resp.  $v$ ] of (2.1) [resp. (2.2)] we mean a function  $u \in \mathcal{D}_q(G)$  [resp.  $v \in \mathcal{D}_Q(G)$ ] which satisfies (2.1) [resp. (2.2)] in  $G$ .

**Proposition 2.1.** *Elliptic inequalities (2.1) and (2.2) are half-linear in the sense that if  $u$  and  $v$  are solutions of (2.1) and (2.2), then  $ku$  and  $k v$  are also solutions of (2.1) and (2.2) for any constant  $k$ , respectively.*

PROOF. It suffices to show that (2.1) is half-linear. Let  $u$  be any solution of (2.1), and  $k(\neq 0)$  be any constant. It is easy to see that

$$\begin{aligned} q[ku] &= \nabla \cdot (|k|^{\alpha(x)-1} k a(x) |\nabla u|^{\alpha(x)-1} \nabla u) \\ &\quad - a(x) (|k|^{\alpha(x)-1} k) (\log(|k||u|)) |\nabla u|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla u \\ &\quad + (|k|^{\alpha(x)-1} k) |\nabla u|^{\alpha(x)-1} b(x) \cdot \nabla u \\ &\quad + (|k|^{\alpha(x)-1} k) c(x) |u|^{\alpha(x)-1} u. \end{aligned} \quad (2.3)$$

A simple computation shows that

$$\begin{aligned} &\nabla \cdot (|k|^{\alpha(x)-1} k a(x) |\nabla u|^{\alpha(x)-1} \nabla u) \\ &= \nabla (|k|^{\alpha(x)-1} k) \cdot (a(x) |\nabla u|^{\alpha(x)-1} \nabla u) \\ &\quad + |k|^{\alpha(x)-1} k \nabla \cdot (a(x) |\nabla u|^{\alpha(x)-1} \nabla u). \end{aligned} \quad (2.4)$$

Since

$$\nabla(|k|^{\alpha(x)-1}k) = |k|^{\alpha(x)-1}k(\log |k|)\nabla\alpha(x),$$

we see that

$$\begin{aligned} & \nabla \cdot (|k|^{\alpha(x)-1}ka(x)|\nabla u|^{\alpha(x)-1}\nabla u) \\ = & a(x)|k|^{\alpha(x)-1}k(\log |k|)|\nabla u|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla u \\ & + |k|^{\alpha(x)-1}k\nabla \cdot (a(x)|\nabla u|^{\alpha(x)-1}\nabla u). \end{aligned} \quad (2.5)$$

Combining (2.3) and (2.5) yields

$$(ku)q[ku] = |k|^{\alpha(x)+1}uq[u] \geq 0$$

for any constant  $k(\neq 0)$ . Since  $(ku)\log |ku| = 0$  for  $k = 0$ , we easily see that  $(ku)q[ku] = 0$  for  $k = 0$ . Hence, we conclude that (2.1) is half-linear.

**Remark 2.1.** We note that (2.1) and (2.2) are half-linear if and only if  $uq[u]$  and  $vQ[v]$  are “homogeneous” functions in  $u$  and  $v$ , respectively, which satisfy

$$\begin{aligned} (ku)q[ku] &= |k|^{\alpha(x)+1}uq[u] \quad (k \in \mathbb{R}), \\ (kv)Q[kv] &= |k|^{\alpha(x)+1}vQ[v] \quad (k \in \mathbb{R}). \end{aligned}$$

**Theorem 2.1 (Picone identity for  $Q$ ).** *If  $v \in \mathcal{D}_Q(G)$  and  $v$  has no zero in  $G$ , then we obtain the following Picone identity for any  $u \in C^1(G; \mathbb{R})$ :*

$$\begin{aligned} & -\nabla \cdot \left( u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) \\ = & -A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \\ & + C(x)|u|^{\alpha(x)+1} \\ & + A(x) \left[ \left| \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla\alpha(x) - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\ & \quad \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\ & \quad \left. - (\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left( \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla\alpha(x) \right. \right. \\ & \quad \left. \left. - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right) \cdot \left( \frac{u}{v} \nabla v \right) \right] \\ & - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}} (vQ[v]) \quad \text{in } G, \end{aligned} \quad (2.6)$$

where  $\varphi(u) = |u|^{\alpha(x)-1}u = |u(x)|^{\alpha(x)-1}u(x)$ .

PROOF. A direct calculation yields

$$\begin{aligned}
& -\nabla \cdot \left( u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \right) \\
= & -\nabla(u\varphi(u)) \cdot \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \\
& -u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla \left( \frac{1}{\varphi(v)} \right) \cdot \nabla v \\
& -\frac{u\varphi(u)}{\varphi(v)}\nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1}\nabla v). \tag{2.7}
\end{aligned}$$

We easily see that

$$\nabla(u\varphi(u)) = (\alpha(x) + 1)\varphi(u)\nabla u + u\varphi(u)(\log |u|)\nabla\alpha(x), \tag{2.8}$$

$$\nabla \left( \frac{1}{\varphi(v)} \right) = -\frac{\alpha(x)}{v\varphi(v)}\nabla v - \frac{\log |v|}{\varphi(v)}\nabla\alpha(x) \tag{2.9}$$

in view of the fact that

$$\nabla\varphi(v) = \alpha(x)\frac{\varphi(v)}{v}\nabla v + (\log |v|)\varphi(v)\nabla\alpha(x).$$

Hence, we observe from (2.8) and (2.9) that

$$\begin{aligned}
& \nabla(u\varphi(u)) \cdot \frac{A(x)|\nabla v|^{\alpha(x)-1}\nabla v}{\varphi(v)} \\
= & (\alpha(x) + 1)\frac{\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha(x)-1}\nabla u \cdot \nabla v \\
& + u\varphi(u)(\log |u|)\frac{A(x)|\nabla v|^{\alpha(x)-1}}{\varphi(v)}\nabla\alpha(x) \cdot \nabla v \\
= & (\alpha(x) + 1)A(x) \left| \frac{u}{v}\nabla v \right|^{\alpha(x)-1} (\nabla u) \cdot \left( \frac{u}{v}\nabla v \right) \\
& + A(x)u(\log |u|)\frac{\varphi(u)}{\varphi(v)}|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v \tag{2.10}
\end{aligned}$$

and

$$\begin{aligned}
& u\varphi(u)A(x)|\nabla v|^{\alpha(x)-1}\nabla \left( \frac{1}{\varphi(v)} \right) \cdot \nabla v \\
= & -\alpha(x)\frac{u\varphi(u)}{v\varphi(v)}A(x)|\nabla v|^{\alpha(x)+1} \\
& -\frac{u\varphi(u)}{\varphi(v)}A(x)(\log |v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v \\
= & -A(x)\alpha(x) \left| \frac{u}{v}\nabla v \right|^{\alpha(x)+1} \\
& -\frac{u\varphi(u)}{\varphi(v)}A(x)(\log |v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x) \cdot \nabla v. \tag{2.11}
\end{aligned}$$

It follows from (1.2) that

$$\begin{aligned}
& \frac{u\varphi(u)}{\varphi(v)} \nabla \cdot (A(x)|\nabla v|^{\alpha(x)-1} \nabla v) \\
&= \frac{u\varphi(u)}{\varphi(v)} \left( Q[v] + A(x)(\log |v|)|\nabla v|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla v \right. \\
&\quad \left. - |\nabla v|^{\alpha(x)-1} B(x) \cdot \nabla v - C(x)|v|^{\alpha(x)-1} v \right) \\
&= \frac{u\varphi(u)}{\varphi(v)} Q[v] + \frac{u\varphi(u)}{\varphi(v)} A(x)(\log |v|)|\nabla v|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla v \\
&\quad - \frac{u\varphi(u)}{\varphi(v)} |\nabla v|^{\alpha(x)-1} B(x) \cdot \nabla v - C(x)|u|^{\alpha(x)+1}. \tag{2.12}
\end{aligned}$$

Combining (2.7), (2.10)–(2.12), we arrive at

$$\begin{aligned}
& -\nabla \cdot \left( u\varphi(u) \frac{A(x)|\nabla v|^{\alpha(x)-1} \nabla v}{\varphi(v)} \right) \\
&= C(x)|u|^{\alpha(x)+1} \\
&\quad + A(x) \left[ \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} - (\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} (\nabla u) \cdot \left( \frac{u}{v} \nabla v \right) \right] \\
&\quad - A(x)u(\log |u|) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} (\nabla \alpha(x)) \cdot \left( \frac{u}{v} \nabla v \right) \\
&\quad + u \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} B(x) \cdot \left( \frac{u}{v} \nabla v \right) \\
&\quad - \frac{u}{\varphi(v)} (\varphi(u)Q[v]) \\
&= C(x)|u|^{\alpha(x)+1} \\
&\quad + A(x) \left[ \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\
&\quad \quad \left. - (\alpha(x)+1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left( \nabla u + \frac{u \log |u|}{\alpha(x)+1} \nabla \alpha(x) \right. \right. \\
&\quad \quad \quad \left. \left. - \frac{u}{(\alpha(x)+1)A(x)} B(x) \right) \cdot \left( \frac{u}{v} \nabla v \right) \right] \\
&\quad - \frac{u\varphi(u)}{v\varphi(v)} (vQ[v]),
\end{aligned}$$

which is equivalent to the desired identity (2.6).

Now we consider the first-order differential system

$$\nabla w = H(x), \tag{2.13}$$

where  $H(x) = (h_1(x), h_2(x), \dots, h_n(x))$  is a vector function of class  $C^1$ , and we define the sequence of functions  $\{g_k(x)\}_{k=1}^n$  by

$$g_1(x) = \int h_1(x) dx_1, \tag{2.14}$$

$$g_k(x) = g_{k-1}(x) + \int \left( h_k(x) - \frac{\partial}{\partial x_k} g_{k-1}(x) \right) dx_k \quad (k = 2, 3, \dots, n). \quad (2.15)$$

**Proposition 2.2.** *The system (2.13) has a  $C^1$ -solution if and only if*

$$\frac{\partial}{\partial x_{j-1}} \left( h_k(x) - \frac{\partial}{\partial x_k} g_{k-1}(x) \right) = 0 \quad (j = 2, 3, \dots, k; k = 2, 3, \dots, n). \quad (2.16)$$

*Then any  $C^1$ -solution  $w$  of (2.13) has the form*

$$w = g_n(x) + C_n \quad (2.17)$$

*for some constant  $C_n$ .*

PROOF. Assume that (2.13) has a  $C^1$ -solution  $w$ , then we obtain

$$\frac{\partial w}{\partial x_1} = h_1(x)$$

and

$$\begin{aligned} w &= \int h_1(x) dx_1 + C_1(x_2, \dots, x_n) \\ &= g_1(x) + C_1(x_2, \dots, x_n) \end{aligned}$$

for some function  $C_1(x_2, \dots, x_n)$ . Since we have

$$\frac{\partial w}{\partial x_2} = h_2(x),$$

we find that  $C_1(x_2, \dots, x_n)$  must satisfy

$$\frac{\partial C_1}{\partial x_2} = h_2(x) - \frac{\partial}{\partial x_2} g_1(x).$$

It is necessary that

$$\frac{\partial}{\partial x_1} \left( h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) = 0$$

and we obtain

$$C_1 = \int \left( h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) dx_2 + C_2(x_3, \dots, x_n)$$

for some function  $C_2(x_3, \dots, x_n)$ , and hence

$$\begin{aligned} w &= g_1(x) + \int \left( h_2(x) - \frac{\partial}{\partial x_2} g_1(x) \right) dx_2 + C_2(x_3, \dots, x_n) \\ &= g_2(x) + C_2(x_3, \dots, x_n). \end{aligned}$$

Repeating the above procedure, we observe that (2.15) is necessary that the solution  $w$  can be written in the form (2.17). It can be shown from the above consideration that the condition (2.16) is sufficient for (2.13) to have a  $C^1$ -solution.

**Theorem 2.2 (Picone identity for  $q$  and  $Q$ ).** Let  $\alpha(x) \in C^2(G; (0, \infty))$  and  $b(x)/a(x) \in C^1(G; \mathbb{R}^n)$ . Assume that  $u \in C^1(G; \mathbb{R})$ ,  $u$  has no zero in  $G$ , and that:

(H<sub>1</sub>) there is a function  $f \in C(\overline{G}; \mathbb{R})$  such that  $f \in C^1(G; \mathbb{R})$  and

$$\nabla f = \frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1)a(x)} \quad \text{in } G.$$

If  $e^f u \in \mathcal{D}_q(G)$ ,  $v \in \mathcal{D}_Q(G)$  and  $v$  has no zero in  $G$ , then we obtain the following Picone identity:

$$\begin{aligned} & \nabla \cdot \left( e^{-(\alpha(x)+1)f} (e^f u) a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) \right. \\ & \quad \left. - \frac{u \varphi(u)}{\varphi(v)} A(x) |\nabla v|^{\alpha(x)-1} \nabla v \right) \\ = & a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)a(x)} b(x) \right|^{\alpha(x)+1} \\ & - A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \\ & + (C(x) - c(x)) |u|^{\alpha(x)+1} \\ & + A(x) \left[ \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\ & \quad \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\ & \quad \left. - (\alpha(x) + 1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left( \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) \right. \right. \\ & \quad \quad \left. \left. - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right) \cdot \left( \frac{u}{v} \nabla v \right) \right] \\ & + e^{-(\alpha(x)+1)f} (e^f u) q[e^f u] - \frac{|u|^{\alpha(x)+1}}{|v|^{\alpha(x)+1}} (v Q[v]) \quad \text{in } G. \end{aligned} \quad (2.18)$$

PROOF. A direct calculation shows that

$$\begin{aligned} & \nabla \cdot \left( e^{-(\alpha(x)+1)f} (e^f u) a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) \right) \\ = & (e^f u) \nabla(e^{-(\alpha(x)+1)f}) \cdot \left( a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) \right) \\ & + e^{-(\alpha(x)+1)f} \nabla(e^f u) \cdot \left( a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) \right) \\ & + e^{-(\alpha(x)+1)f} (e^f u) \nabla \cdot \left( a(x) |\nabla(e^f u)|^{\alpha(x)-1} \nabla(e^f u) \right). \end{aligned} \quad (2.19)$$

Since

$$\nabla(e^{-(\alpha(x)+1)f}) = e^{-(\alpha(x)+1)f} \left( -(\nabla \alpha(x)) f - (\alpha(x) + 1) \nabla f \right),$$



we observe, using the hypothesis (H<sub>1</sub>), that

$$\begin{aligned}
& (e^f u) \nabla (e^{-(\alpha(x)+1)f}) \\
&= e^{-(\alpha(x)+1)f} \left[ -(e^f u) (\nabla \alpha(x)) f - (\alpha(x) + 1) e^f u \nabla f \right] \\
&= e^{-(\alpha(x)+1)f} \left[ -(e^f u) (\nabla \alpha(x)) f - e^f (u \log |u|) \nabla \alpha(x) + e^f \frac{u}{a(x)} b(x) \right] \\
&= e^{-(\alpha(x)+1)f} (e^f u) \left[ -\log |e^f u| \nabla \alpha(x) + \frac{b(x)}{a(x)} \right]
\end{aligned}$$

and therefore

$$\begin{aligned}
& (e^f u) \nabla (e^{-(\alpha(x)+1)f}) \cdot \left( a(x) |\nabla (e^f u)|^{\alpha(x)-1} \nabla (e^f u) \right) \\
&= e^{-(\alpha(x)+1)f} (e^f u) \left[ -a(x) \log |e^f u| |\nabla (e^f u)|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla (e^f u) \right. \\
&\quad \left. + |\nabla (e^f u)|^{\alpha(x)-1} b(x) \cdot \nabla (e^f u) \right]. \tag{2.20}
\end{aligned}$$

It is clear that

$$\begin{aligned}
& e^{-(\alpha(x)+1)f} \nabla (e^f u) \cdot a(x) |\nabla (e^f u)|^{\alpha(x)-1} \nabla (e^f u) \\
&= e^{-(\alpha(x)+1)f} a(x) |\nabla (e^f u)|^{\alpha(x)+1} \\
&= a(x) |e^{-f} \nabla (e^f u)|^{\alpha(x)+1} \\
&= a(x) |\nabla u + u \nabla f|^{\alpha(x)+1} \\
&= a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1) a(x)} b(x) \right|^{\alpha(x)+1} \tag{2.21}
\end{aligned}$$

in view of the hypothesis (H<sub>1</sub>). From (2.19)–(2.21) it follows that

$$\begin{aligned}
& \nabla \cdot \left( e^{-(\alpha(x)+1)f} (e^f u) a(x) |\nabla (e^f u)|^{\alpha(x)-1} \nabla (e^f u) \right) \\
&= e^{-(\alpha(x)+1)f} (e^f u) \left[ \nabla \cdot \left( a(x) |\nabla (e^f u)|^{\alpha(x)-1} \nabla (e^f u) \right) \right. \\
&\quad \left. - a(x) \log |e^f u| |\nabla (e^f u)|^{\alpha(x)-1} \nabla \alpha(x) \cdot \nabla (e^f u) \right. \\
&\quad \left. + |\nabla (e^f u)|^{\alpha(x)-1} b(x) \cdot \nabla (e^f u) \right] \\
&\quad + a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1) a(x)} b(x) \right|^{\alpha(x)+1} \\
&= a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1) a(x)} b(x) \right|^{\alpha(x)+1} \\
&\quad + e^{-(\alpha(x)+1)f} (e^f u) \left[ q[e^f u] - c(x) |e^f u|^{\alpha(x)-1} e^f u \right] \\
&= a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1) a(x)} b(x) \right|^{\alpha(x)+1} \\
&\quad - c(x) |u|^{\alpha(x)+1} + e^{-(\alpha(x)+1)f} (e^f u) q[e^f u]. \tag{2.22}
\end{aligned}$$

Combining (2.6) with (2.22) yields the desired identity (2.18).

**Remark 2.2.** In order to explain the role of the function  $f$  in the hypothesis  $(H_1)$ , we treat the ordinary differential operator  $\ell$  and the variation  $V[y]$  defined by

$$\begin{aligned}\ell[y] &= (a(t)y')' + b(t)y', \quad t \in (t_1, t_2), \\ V[y] &= \int_{t_1}^{t_2} a(t) \left| y' - \frac{b(t)}{2a(t)}y \right|^2 dt,\end{aligned}$$

where  $a(t) \in C^1([t_1, t_2]; (0, \infty))$  and  $b(t) \in C([t_1, t_2]; \mathbb{R})$ . Letting

$$f(t) = - \int \frac{b(t)}{2a(t)} dt,$$

we observe that

$$\begin{aligned}V[y] &= \int_{t_1}^{t_2} a(t) \left| e^{-f(t)} (e^{f(t)}y)' \right|^2 dt \\ &= \int_{t_1}^{t_2} e^{-2f(t)} a(t) \left( (e^{f(t)}y)' \right)^2 dt \\ &= - \int_{t_1}^{t_2} e^{-2f(t)} (e^{f(t)}y) \ell[e^{f(t)}y] dt\end{aligned}$$

if  $y(t_1) = y(t_2) = 0$ . Introducing the function  $f(t)$ , we can consider the function  $e^{f(t)}y$  to be a new unknown function.

**Remark 2.3.** We give an example which illustrates the hypothesis  $(H_1)$ . Let  $n = 1$ ,  $G = (0, \pi)$ ,  $u = \sin x$ ,  $\alpha(x) = e^{\sin x + 1} - 1$ ,  $a(x) = 1$ ,  $b(x) = -(\cos x)e^{\sin x + 1}$ . Defining  $f(x)$  by

$$f(x) = \begin{cases} (\sin x) \log \sin x, & x \in (0, \pi) \\ 0 & \text{at } x = 0, \pi, \end{cases}$$

we conclude that

$$\begin{aligned}f'(x) &= (\cos x) \log \sin x + \cos x \\ &= \frac{\log |u|}{\alpha(x) + 1} \alpha'(x) - \frac{b(x)}{(\alpha(x) + 1)a(x)} \quad \text{in } (0, \pi).\end{aligned}$$

Moreover, we see that  $f(x)$  is a continuous function on  $[0, \pi]$  in view of the fact that  $\lim_{\varepsilon \rightarrow +0} \varepsilon \log \varepsilon = 0$ .

**Remark 2.4.** It follows from Proposition 2.2 that if  $(H_1)$  holds, then the function

$$\frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1)a(x)} \quad (2.23)$$

must satisfy (2.16) in  $G$  with  $H(x)$  replaced by (2.23). It is necessary that  $\alpha(x) \in C^2$  and  $b(x)/a(x) \in C^1$ . For example, we treat the case where  $n = 2$ ,  $G = (0, \pi) \times (0, \pi)$ ,  $u = \sin x_1 \sin x_2$ ,  $\alpha(x) = e^{\sin x_1 \sin x_2 + 1} - 1$ ,  $a(x) = 1$ , and

$$b(x) = (-(\cos x_1 \sin x_2)e^{\sin x_1 \sin x_2 + 1}, -(\sin x_1 \cos x_2)e^{\sin x_1 \sin x_2 + 1}).$$

Then we have

$$\frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1)a(x)} = (h_1(x_1, x_2), h_2(x_1, x_2)),$$

where

$$\begin{aligned} h_1(x_1, x_2) &= (\cos x_1 \log \sin x_1 + \cos x_1) \sin x_2 + \cos x_1 \sin x_2 \log \sin x_2, \\ h_2(x_1, x_2) &= (\cos x_2 \log \sin x_2 + \cos x_2) \sin x_1 + \cos x_2 \sin x_1 \log \sin x_1. \end{aligned}$$

It is easy to check that

$$\frac{\partial}{\partial x_1} \left( h_2(x_1, x_2) - \frac{\partial}{\partial x_2} \int h_1(x_1, x_2) dx_1 \right) = 0,$$

and the solution  $f$  of

$$\nabla f = (h_1(x_1, x_2), h_2(x_1, x_2)) \quad \text{in } G$$

is written in the form

$$f = (\sin x_1 \log \sin x_1) \sin x_2 + (\sin x_2 \log \sin x_2) \sin x_1,$$

which is continuous on  $\overline{G} = [0, \pi] \times [0, \pi]$  by defining  $f = 0$  on  $\partial G$ .

### 3. Sturmian comparison theorems

On the basis of the Picone identity in Section 2 we present Sturmian comparison theorems for the half-linear elliptic operators  $q$  and  $Q$ .

**Lemma 3.1.** *The inequality*

$$|\xi|^{\alpha(x)+1} + \alpha(x) |\eta|^{\alpha(x)+1} - (\alpha(x) + 1) |\eta|^{\alpha(x)-1} \xi \cdot \eta \geq 0 \quad (3.1)$$

is valid for  $x \in G$ ,  $\xi, \eta \in \mathbb{R}^n$ , where the equality holds if and only if  $\xi = \eta$ .

PROOF. For any fixed  $x \in G$ , the inequality (3.1) holds for any  $\xi, \eta \in \mathbb{R}^n$  by Hardy, Littlewood and Pólya [17, Theorem 41] and Kusano, Jaroš and Yoshida [19, Lemma 2.1].

**Theorem 3.1 (Sturmian comparison theorem).** *Let  $\alpha(x) \in C^2(G; (0, \infty))$  and  $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$ . Assume that there exists a function  $u \in C^1(\overline{G}; \mathbb{R})$  such that  $u = 0$  on  $\partial G$ ,  $u$  has no zero in  $G$ , the hypothesis  $(H_1)$  of Theorem 2.2 holds and that:*

$(H_2)$  *there is a function  $F \in C(\overline{G}; \mathbb{R})$  such that  $F \in C^1(G; \mathbb{R})$  and*

$$\nabla F = \frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{B(x)}{(\alpha(x) + 1)A(x)} \quad \text{in } G.$$

If the following conditions are satisfied:

(i)  $e^f u \in \mathcal{D}_q(G)$  and

$$(e^f u)q[e^f u] \geq 0 \quad \text{in } G;$$

(ii)

$$\begin{aligned} V_G[u] := & \int_G \left[ a(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)a(x)} b(x) \right|^{\alpha(x)+1} \right. \\ & \left. - A(x) \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\ & \left. + (C(x) - c(x)) |u|^{\alpha(x)+1} \right] dx \geq 0, \end{aligned}$$

then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\overline{G}$ .

PROOF. Suppose to the contrary that there exists a solution  $v \in \mathcal{D}_Q(G)$  of (2.2) such that  $v$  has no zero on  $\overline{G}$ . Integrating the Picone identity (2.18) over  $G$  and using the divergence theorem, we obtain

$$0 \geq V_G[u] + \int_G W(u, v) dx \geq 0,$$

which yields the following

$$\int_G W(u, v) dx = 0,$$

where

$$\begin{aligned} W(u, v) := & A(x) \left[ \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \right. \\ & \left. + \alpha(x) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)+1} \right. \\ & \left. - (\alpha(x) + 1) \left| \frac{u}{v} \nabla v \right|^{\alpha(x)-1} \left( \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) \right. \right. \\ & \left. \left. - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right) \cdot \left( \frac{u}{v} \nabla v \right) \right]. \end{aligned}$$

It follows from Lemma 3.1 that

$$\nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \equiv \frac{u}{v} \nabla v \quad \text{in } G,$$

that is,

$$\nabla u + u \nabla F \equiv \frac{u}{v} \nabla v \quad \text{in } G,$$

from which we have

$$e^{-F}v\nabla\left(e^F\frac{u}{v}\right)\equiv 0 \quad \text{in } G.$$

Therefore, there exists a constant  $k_0$  such that  $e^Fu/v = k_0$  in  $G$  and hence on  $\overline{G}$  by continuity. Since  $u = 0$  on  $\partial G$ , we see that  $k_0 = 0$ , which contradicts the hypothesis that  $u$  is nontrivial. The proof is complete.

**Corollary 3.1.** *Let  $\alpha(x) \in C^2(G; (0, \infty))$ ,  $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$ . Assume that:*

- (i)  $\frac{b(x)}{a(x)} = \frac{B(x)}{A(x)} \quad \text{in } G;$
- (ii)  $a(x) \geq A(x), \quad C(x) \geq c(x) \quad \text{in } G.$

*If there exists a function  $u \in C^1(\overline{G}; \mathbb{R})$  such that  $u = 0$  on  $\partial G$ ,  $u$  has no zero in  $G$ , the hypothesis  $(H_1)$  of Theorem 2.2 holds and (i) of Theorem 3.1 is satisfied, then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\overline{G}$ .*

PROOF. The conditions (i), (ii) imply that  $V_G[u] \geq 0$  for any  $u \in C^1(\overline{G}; \mathbb{R})$  and  $(H_2)$  is the same as  $(H_1)$ . The conclusion follows from Theorem 3.1.

**Corollary 3.2.** *Let  $\alpha(x) \in C^2(G; (0, \infty))$ ,  $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$ . Assume that the hypotheses (i), (ii) of Corollary 3.1 are satisfied, and that there exists a nontrivial function  $u \in C^1(\overline{G}; \mathbb{R})$  which satisfies  $u = 0$  on  $\partial G$  and the following:*

*$(\tilde{H}_1)$  there is a function  $f \in C(\overline{G}; \mathbb{R})$  such that  $f \in C^1(N_u; \mathbb{R})$  and*

$$\nabla f = \frac{\log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{b(x)}{(\alpha(x) + 1)a(x)} \quad \text{in } N_u,$$

*where*

$$N_u := \{x \in G; u(x) \neq 0\}.$$

*If  $e^fu \in \mathcal{D}_q(N_u)$ ,  $(e^fu)q[e^fu] \geq 0$  in  $N_u$ , then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\overline{G}$ .*

PROOF. Since  $u$  is nontrivial and  $u = 0$  on  $\partial G$ , there is a domain  $G_0 \subset G$  for which  $u = 0$  on  $\partial G_0$  and  $u$  has no zero in  $G_0$ . Applying Corollary 3.1 with  $G$  replaced by  $G_0$ , we conclude that every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\overline{G_0} \subset \overline{G}$ , that is,  $v$  has a zero on  $\overline{G}$ .

Next we deal with the case where  $G$  is the annular domain  $A(r_1, r_2)$  defined by

$$A(r_1, r_2) = \{x \in \mathbb{R}^n; r_1 < |x| < r_2\} \quad (r_1 < r_2).$$

We use the notation:

$$\begin{aligned} A[r_1, r_2] &= \{x \in \mathbb{R}^n; r_1 \leq |x| \leq r_2\}, \\ S_r &= \{x \in \mathbb{R}^n; |x| = r\}. \end{aligned}$$

Let  $\bar{A}(r)$  and  $\bar{C}(r)$  denote the spherical means of  $A(x)$  and  $C(x)$  over the sphere  $S_r$ , respectively, that is,

$$\begin{aligned}\bar{A}(r) &= \frac{1}{\omega_n r^{n-1}} \int_{S_r} A(x) dS = \frac{1}{\omega_n} \int_{S_1} A(r, \theta) d\omega, \\ \bar{C}(r) &= \frac{1}{\omega_n r^{n-1}} \int_{S_r} C(x) dS = \frac{1}{\omega_n} \int_{S_1} C(r, \theta) d\omega,\end{aligned}$$

where  $\omega_n$  is the surface area of the unit sphere  $S_1$ ,  $(r, \theta)$  is the hyperspherical coordinates in  $\mathbb{R}^n$  and  $\omega$  is the measure on  $S_1$ .

We assume that:

$$(H_3) \quad \alpha(x) \equiv \alpha(|x|) \quad \text{in } A(r_1, r_2);$$

$$(H_4) \quad \frac{B(x)}{A(x)} = B_0(|x|) \frac{x_i}{|x|} \quad \text{in } A(r_1, r_2) \text{ for some function } B_0(r) \in C[r_1, r_2].$$

Associated with (2.2) we treat the half-linear elliptic operator  $\tilde{q}$  defined by

$$\begin{aligned}\tilde{q}[u] &:= \nabla \cdot \left( \bar{A}(|x|) |\nabla u|^{\alpha(|x|)-1} \nabla u \right) - \bar{A}(|x|) (\log |u|) |\nabla u|^{\alpha(|x|)-1} \nabla \alpha(|x|) \cdot \nabla u \\ &\quad + |\nabla u|^{\alpha(|x|)-1} \bar{A}(|x|) B_0(|x|) \frac{x_i}{|x|} \cdot \nabla u + \bar{C}(|x|) |u|^{\alpha(|x|)-1} u.\end{aligned}$$

We define the half-linear ordinary differential operator  $q_0$  by

$$\begin{aligned}q_0[y] &:= \left( r^{n-1} \bar{A}(r) |y'|^{\alpha(r)-1} y' \right)' - r^{n-1} \bar{A}(r) (\log |y|) |y'|^{\alpha(r)-1} \alpha'(r) y' \\ &\quad + r^{n-1} \bar{A}(r) B_0(r) |y'|^{\alpha(r)-1} y' + r^{n-1} \bar{C}(r) |y|^{\alpha(r)-1} y,\end{aligned}$$

and the domain  $\mathcal{D}_{q_0}((r_1, r_2))$  of  $q_0$  is defined to be the set of all functions  $y$  of class  $C^1[r_1, r_2]$  such that  $r^{n-1} \bar{A}(r) |y'|^{\alpha(r)-1} y' \in C^1(r_1, r_2) \cap C[r_1, r_2]$ . If  $y(r)$  is a solution of  $y q_0[y] \geq 0$ , then  $u(x) = y(|x|)$  is a radially symmetric solution of  $u \tilde{q}[u] \geq 0$ .

**Theorem 3.2.** *Assume that the hypotheses  $(H_3)$ ,  $(H_4)$  hold. If there exists a function  $z = z(r) \in \mathcal{D}_{q_0}((r_1, r_2))$  such that:*

$$(i) \quad z(r_1) = z(r_2) = 0 \text{ and } z(r) > 0 \text{ in } (r_1, r_2);$$

$$(ii) \quad \text{there is a function } f_0 = f_0(r) \in C[r_1, r_2] \text{ such that } f_0 \in C^1(r_1, r_2) \text{ and}$$

$$f'_0(r) = \frac{\log |z(r)|}{\alpha(r) + 1} \alpha'(r) - \frac{B_0(r)}{\alpha(r) + 1} \quad \text{in } (r_1, r_2);$$

$$(iii) \quad e^{f_0} z \in \mathcal{D}_{q_0}((r_1, r_2)) \text{ and}$$

$$(e^{f_0} z) q_0[e^{f_0} z] \geq 0 \quad \text{in } (r_1, r_2),$$

then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\bar{G}$ .

PROOF. Suppose on the contrary, that there is a solution  $v \in \mathcal{D}_Q(G)$  of (2.2) such that  $v$  has no zero on  $\bar{G}$ . Defining

$$u(x) := z(|x|),$$

we compare  $u\tilde{q}[u] \geq 0$  with (2.2). The condition (ii) implies that the hypotheses  $(H_1)$ ,  $(H_2)$  are satisfied for the case where  $u(x) = z(|x|)$ ,  $f = F = f_0(|x|)$  and

$$\bar{A}(|x|)B_0(|x|)\frac{x_i}{|x|} / \bar{A}(|x|) = B(x)/A(x) = B_0(|x|)\frac{x_i}{|x|}.$$

Noting that

$$\begin{aligned} & \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)\bar{A}(|x|)} \bar{A}(|x|)B_0(|x|)\frac{x_i}{|x|} \right|^{\alpha(x)+1} \\ &= \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{(\alpha(x) + 1)A(x)} B(x) \right|^{\alpha(x)+1} \\ &= \left| z'(r) + \frac{z(r) \log |z(r)|}{\alpha(r) + 1} \alpha'(r) - \frac{z(r)}{\alpha(r) + 1} B_0(r) \right|^{\alpha(r)+1} \quad \text{on } S_r, \end{aligned}$$

we easily arrive at

$$\begin{aligned} & V_{A(r_1, r_2)}[u] \\ &= \int_{A(r_1, r_2)} \left[ (\bar{A}(|x|) - A(x)) \times \right. \\ & \quad \times \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{\alpha(x) + 1} B_0(|x|)\frac{x_i}{|x|} \right|^{\alpha(x)+1} \\ & \quad \left. + (C(x) - \bar{C}(|x|))|u|^{\alpha(x)+1} \right] dx \\ &= \int_{r_1}^{r_2} \int_{S_r} \left[ (\bar{A}(|x|) - A(x)) \times \right. \\ & \quad \times \left| \nabla u + \frac{u \log |u|}{\alpha(x) + 1} \nabla \alpha(x) - \frac{u}{\alpha(x) + 1} B_0(|x|)\frac{x_i}{|x|} \right|^{\alpha(x)+1} \\ & \quad \left. + (C(x) - \bar{C}(|x|))|u|^{\alpha(x)+1} \right] dS dr \\ &= \int_{r_1}^{r_2} \int_{S_1} \left[ (\bar{A}(r) - A(r, \theta)) \times \right. \\ & \quad \times \left| z'(r) + \frac{z(r) \log |z(r)|}{\alpha(r) + 1} \alpha'(r) - \frac{z(r)}{\alpha(r) + 1} B_0(r) \right|^{\alpha(r)+1} \\ & \quad \left. + (C(r, \theta) - \bar{C}(r))|z(r)|^{\alpha(r)+1} \right] r^{n-1} d\omega dr \end{aligned}$$

$$\begin{aligned}
&= \omega_n \int_{r_1}^{r_2} \left[ \left( \bar{A}(r) - \frac{1}{\omega_n} \int_{S_1} A(r, \theta) d\omega \right) \times \right. \\
&\quad \times \left| z'(r) + \frac{z(r) \log |z(r)|}{\alpha(r) + 1} \alpha'(r) - \frac{z(r)}{\alpha(r) + 1} B_0(r) \right|^{\alpha(r)+1} \\
&\quad \left. + \left( \frac{1}{\omega_n} \int_{S_1} C(r, \theta) d\omega - \bar{C}(r) \right) |z(r)|^{\alpha(r)+1} \right] r^{n-1} dr \\
&= 0.
\end{aligned}$$

Therefore, all hypotheses of Theorem 3.1 are satisfied, and the conclusion follows from Theorem 3.1. The proof is complete.

#### 4. Specializations

In this Section we give some specializations to the case where  $\alpha(x) = \alpha > 0$ , and the case where  $n = 1$ ,  $b(x) = B(x) \equiv 0$ .

**Theorem 4.1.** *Let  $\alpha(x) = \alpha > 0$  and  $b(x)/a(x), B(x)/A(x) \in C^1(G; \mathbb{R}^n)$ . Assume that there exists a nontrivial function  $u \in C^1(\bar{G}; \mathbb{R})$  such that  $u = 0$  on  $\partial G$ , and that the following hypotheses are satisfied:*

( $\hat{H}_1$ ) *there is a function  $f \in C(\bar{G}; \mathbb{R})$  such that  $f \in C^1(G; \mathbb{R})$  and*

$$\nabla f = -\frac{b(x)}{(\alpha + 1)a(x)} \quad \text{in } G;$$

( $\hat{H}_2$ ) *there exists a function  $F \in C(\bar{G}; \mathbb{R})$  such that  $F \in C^1(G; \mathbb{R})$  and*

$$\nabla F = -\frac{B(x)}{(\alpha + 1)A(x)} \quad \text{in } G.$$

If  $e^f u \in \mathcal{D}_q(G)$ ,

$$(e^f u)q[e^f u] \geq 0 \quad \text{in } G,$$

and

$$\begin{aligned}
V_G[u] &= \int_G \left[ a(x) \left| \nabla u - \frac{u}{(\alpha + 1)a(x)} b(x) \right|^{\alpha+1} \right. \\
&\quad \left. - A(x) \left| \nabla u - \frac{u}{(\alpha + 1)A(x)} B(x) \right|^{\alpha+1} \right. \\
&\quad \left. + (C(x) - c(x)) |u|^{\alpha+1} \right] dx \geq 0,
\end{aligned}$$

then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\bar{G}$ .



PROOF. Since  $\nabla\alpha(x) \equiv 0$  on  $\overline{G}$ , the Picone identity (2.18) holds without the hypothesis that  $u$  has no zero in  $G$ . Therefore, the conclusion follows from Theorem 3.1.

The following corollary was established by Dunninger [12], Kusano, Jaroš and Yoshida [19].

**Corollary 4.1.** *Let  $\alpha(x) = \alpha > 0$  and  $b(x) = B(x) \equiv 0$  in  $G$ . If there exists a nontrivial function  $u \in \mathcal{D}_q(G)$  such that  $u = 0$  on  $\partial G$ ,  $uq[u] \geq 0$  in  $G$ , and*

$$V_G[u] = \int_G [(a(x) - A(x))|\nabla u|^{\alpha+1} + (C(x) - c(x))|u|^{\alpha+1}] dx \geq 0,$$

*then every solution  $v \in \mathcal{D}_Q(G)$  of (2.2) must vanish at some point of  $\overline{G}$ .*

PROOF. Since  $b(x) = B(x) \equiv 0$  on  $\overline{G}$ , we can choose  $f = F \equiv 0$  on  $\overline{G}$ . Hence, the conclusion follows from Theorem 4.1.

Next we consider the special case where  $n = 1$ ,  $b(x) = B(x) \equiv 0$ , that is, we let  $x_1 = t$ ,  $G = (t_1, t_2)$ , and define  $q_1$  and  $Q_1$  by

$$\begin{aligned} q_1[y] := & \left( a(t)|y'|^{\alpha(t)-1}y' \right)' - a(t)(\log |y|)|y'|^{\alpha(t)-1}\alpha'(t)y' \\ & + c(t)|y|^{\alpha(t)-1}y, \end{aligned} \quad (4.1)$$

$$\begin{aligned} Q_1[z] := & \left( A(t)|z'|^{\alpha(t)-1}z' \right)' - A(t)(\log |z|)|z'|^{\alpha(t)-1}\alpha'(t)z' \\ & + C(t)|z|^{\alpha(t)-1}z, \end{aligned} \quad (4.2)$$

where the coefficients appearing in (4.1) and (4.2) are supposed to satisfy the same conditions as in Section 2. The domains  $\mathcal{D}_{q_1}(I)$ ,  $\mathcal{D}_{Q_1}(I)$  are defined as in Section 2, where  $I = (t_1, t_2)$ .

**Theorem 4.2.** *Let  $\alpha(x) \in C^2(I; (0, \infty)) \cap C^1(\overline{I}; (0, \infty))$ . Assume that there exists a function  $y \in C^1(\overline{I}; \mathbb{R})$  such that  $y(t_1) = y(t_2) = 0$ ,  $y$  has no zero in  $I$ , and the following hypothesis is satisfied:*

*( $\overline{H}_1$ ) there is a function  $f \in C(\overline{I}; \mathbb{R})$  such that  $f \in C^1(I; \mathbb{R})$  and*

$$f'(t) = \frac{\log |y|}{\alpha(t) + 1} \alpha'(t) \quad \text{in } I.$$

*If  $e^f y \in \mathcal{D}_{q_1}(I)$ ,*

$$(e^f y)_{q_1}[e^f y] \geq 0 \quad \text{in } I,$$

*and*

$$\begin{aligned} V_I[u] = & \int_I \left[ (a(t) - A(t)) \left| y' + \frac{y \log |y|}{\alpha(t) + 1} \alpha'(t) \right|^{\alpha(t)+1} \right. \\ & \left. + (C(t) - c(t)) |y|^{\alpha(t)+1} \right] dt \geq 0, \end{aligned}$$

*then every solution  $z \in \mathcal{D}_{Q_1}(I)$  of  $zQ_1[z] \leq 0$  must vanish at some point of  $\overline{I}$ .*

PROOF. The conclusion follows from Theorem 3.1.

We state the analogue of Corollary 3.1.

**Corollary 4.2.** *Let  $\alpha(x) \in C^2(I; (0, \infty)) \cap C^1(\bar{I}; (0, \infty))$ . Assume that there is a function  $y \in C^1(\bar{I}; \mathbb{R})$  such that  $y(t_1) = y(t_2) = 0$ ,  $y$  has no zero in  $I$ , and the hypothesis  $(\bar{H}_1)$  of Theorem 4.2 holds. If  $e^f y \in \mathcal{D}_{q_1}(I)$ ,*

$$(e^f y)_{q_1}[e^f y] \geq 0 \quad \text{in } I,$$

and

$$a(t) \geq A(t), \quad C(t) \geq c(t) \quad \text{in } I,$$

then every solution  $z \in \mathcal{D}_{Q_1}(I)$  of  $z_{Q_1}[z] \leq 0$  must vanish at some point of  $\bar{I}$ .

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